Journal of Statistical Physics, Vol. 47, Nos. 5/6, 1987

Charged Surfaces and the Analyticity Properties of the String Tension in Lattice Gauge Theories

Christian Borgs[1]

Received November 24, 1986; revision received January 14, 1987

A method to prove the fact that the string tension σ in strongly coupled lattice gauge theories is of the form $\sigma = -\log \beta + \tilde{\sigma}$, where $\tilde{\sigma}$ is an analytic function of the inverse coupling $\beta = 1/g^2$, is presented. Its relation to random surface methods, in particular to the work of Debrushin and Holický, Kotecký, and Zahradník, is discussed.

KEY WORDS: Lattice gauge theories; string tension; Wilson loops; random surfaces; walls and ceilings; aggregate models.

It is well known that the expectation values of local observables in strongly coupled lattice gauge theories are analytic functions of the inverse coupling $\beta = 1/g^2$ (see, e.g., Ref. 1). In particular, this is true for the Wilson loop observable W_{tl} corresponding to a closed, rectangular loop C_{tl} of side lengths t, l. This does not imply, however, that the string tension σ, which is obtained from the expectation value $\langle W_{tl} \rangle$ by

$$\sigma = -\lim_{t,l \to \infty} \frac{1}{tl} \log \langle W_{tl} \rangle \tag{1}$$

is analytic in β. Indeed, it is believed that σ is of the form

$$\sigma = -\log \beta + \tilde{\sigma}(\beta) \tag{2}$$

where the function $\tilde{\sigma}(\cdot)$ has an analytic continuation in some neighborhood of zero.

---

[1] Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland.

This paper is based on a talk presented at the conference on Statistical Mechanics of Phase Transitions—Mathematical and Physical Aspects, Trebon, September 1–6, 1986.

0022-4715/87/0600-0867$05.00/0

While $\tilde{\sigma}$ has already been *calculated* up to order 12 in β,[2] no rogorous proof for the assumed analyticity of $\tilde{\sigma}$ has appeared. Note that this cannot be done by simply working out the cluster expansion for $\tilde{\sigma}$ sketched by Münster[2]; in fact, it can be shown[3] that the corresponding expansion is not absolutely summable.[2]

In this paper I present a method which was used recently[3] to prove the analyticity of $\tilde{\sigma}(\cdot)$. It structurally is very similar to the methods used by Holický *et al.*[4] to show the existence of non-translation-invariant Gibbs states in the context of Pirogov–Sinai theory. The organisation of this paper is as follows: I first review Wilson's formulation[5] of lattice gauge theories. Then the usual strong coupling cluster expansion is used to derive a representation of $\langle W_{tl} \rangle$ as a decorated sum over charged surfaces with boundary C_{tl}. The deviations of these surfaces from the minimal surface with boundary C_{tl} are then considered as particles of a gas of excitations living on the minimal surface S_0. For large g^2 this gas turns out to be dilute, and the Mayer expansion for the free energy density of this gas, which is essentially equal to $\tilde{\sigma}$, is convergent. This implies the analyticity of $\tilde{\sigma}$ in a neighborhood of $\beta = 0$.

While I do not present all the details of the proof here, I point out the main differences between it and Ref. 4, and also mention some of the problems that had to be solved in order to complete the proof.

I consider strongly coupled lattice Yang–Mills theories defined in a finite box $\Lambda \subset \mathbb{Z}^d$, where d is the space-time dimension, and the lattice spacing was set to one for notational convenience. As usual, a gauge field configuration is a map from the positively oriented nearest neighbor (n.n.) pairs[3] in Λ into the gauge group G: $\langle xy \rangle \mapsto g_{xy}$; if $\langle yx \rangle$ is positively oriented, one sets $g_{xy} = g_{yx}^{-1}$. To each closed loop C of n.n. pairs in Λ is assigned the loop variable $g_C = \prod_{l \in C} g_l$; in particular, one assigns to each plaquette p in Λ the plaquette variable $g_{\partial p}$, often simply denoted g_p.

Expectation values in the volume Λ are defined by

$$\langle \cdot \rangle_\Lambda = \frac{1}{Z_\Lambda} \int \cdot\, e^{S_\Lambda} \prod_{\langle xy \rangle} dg_{xy}$$

where Z_Λ is chosen such that $\langle 1 \rangle_\Lambda = 1$ and S_Λ is the Wilson action

$$S_\Lambda = \sum_{p \in \Lambda^2} s(g_p), \qquad s(\cdot) = (1/g^2) \operatorname{Re} \operatorname{Tr}(\cdot) - \text{const} \tag{3}$$

[2] This does not affect the calculation $\tilde{\sigma}$ in Ref. 2, because it can be shown by methods similar to those presented in this paper that Münster's expansion is indeed convergent and gives the correct series for $\tilde{\sigma}$ if it is resummed in orders of β.

[3] I sometimes use the word "link" for a n.n. pair.

Here g^2 is the coupling constant, the trace $\mathrm{Tr}(\cdot)$ is taken in an irreducible representation of the gauge group G [for $G = U(N)$ or $SU(N)$ one usually takes the fundamental representation], and the constant is chosen in such a way that

$$\int e^{s(g)}\, dg = 1 \tag{4}$$

Λ^2 denotes the set of positively oriented plaquettes in Λ. The thermodynamic limit $\Lambda \to \mathbb{Z}^d$ of $\langle \cdot \rangle_\Lambda$, which exists due to the convergence of the strong coupling cluster expansion, is denoted $\langle \cdot \rangle$.

Let Σ be a plane spanned by two of the coordinate directions, and let C_{tl} be a rectangular loop of side length t, resp. l, in Σ. The Wilson loop observable W_{tl} corresponding to C_{1l} is then defined as

$$W_{tl} = \mathrm{Tr}\, g_{C_{tl}}$$

where I assume that the trace is taken in the same representation as in the action. The string tension σ is now defined by Eq. (1):

$$\sigma := -\lim_{t,l \to \infty} \frac{1}{tl} \log \langle W_{tl} \rangle$$

Since $\sigma = 0$ for representations that are trivial on the center[4] $Z(G)$, I assume that the representation q_C that was chosen to define W_{tl} represents $Z(G)$ in a nontrivial way.

Let Γ_0 be the set of finite, nonempty, connected sets of positively oriented plaquettes, $\rho(\cdot) = e^{s(\cdot)} - 1$, and define

$$z(\gamma) = \int \prod_{p \in \gamma} \rho(g_p) \prod_{\langle xy \rangle} dg_{xy}$$
$$z_C(\gamma) = \int W_{tl} \prod_{p \in \gamma} \rho(g_p) \prod_{\langle xy \rangle} dg_{xy} \tag{5}$$

The strong coupling cluster expansion then gives the following representation for $\langle W_{tl} \rangle$:

$$\langle W_{tl} \rangle = \sum_{n=0}^{\infty} \sum_{\gamma_0, \dots, \gamma_n \in \Gamma_0} z_C(\gamma_0) \frac{\phi_c(\gamma_0, \dots, \gamma_n)}{n!} \prod_{i=1}^{n} z(\gamma_i) \tag{6}$$

[4] This fact is stated as a theorem at various places in the literature, see, e.g., Ref. 1. To my knowledge, however, there is no proof in the literature that is correct or could be extended to a correct proof with some extra work. A correct proof will be sketched at the end of this paper.

Here the sum is over finite sequences $\gamma_0,..., \gamma_n$ in Γ_0 (where some of the γ_i might be equal), γ_0 has to be connected to C_{tl}, and $\phi_c(\gamma_0,..., \gamma_n)$ is a combinatoric coefficient invariant under permutations of $\gamma_0,..., \gamma_n$. It is zero if $\gamma_0 \cup \cdots \cup \gamma_n$ is not a connected set. For large g^2 the expansion (6) converges absolutely (see, e.g., Ref. 1).

At this point it can already be seen why (6) is a representation of $\langle W_{tl} \rangle$ as a sum over "decorated surfaces with boundary C_{tl}." Let us first consider the simple case where the gauge group G is Z_2. In this case it is easy to see that $z_C(\gamma)=0$ unless $\partial\gamma = C_{tl}$ and $z(\gamma)=0$ unless $\partial\gamma=\varnothing$ ($\partial\gamma$ denotes the set of n.n. pairs in $\mathbb{Z}^d$, which belong to an odd number of plaquettes in γ). Therefore, (6) is an expansion of $\langle W_{tl} \rangle$ into a sum over surfaces γ_0 with boundary C_{tl}, decorated with clusters of closed surfaces (I will make the notion of a decoration of γ_0 more precise below.) A typical contribution is shown in Fig. 1.

For a general group γ_0 is not necessarily a surface with boundary C_{tl} in the geometrical sense. It can, however, be shown[1] that $z_C(\gamma_0)=0$ if there is no Γ-valued 2-form ω defined on γ_0, where Γ is the dual of the center of G, such that $d^*\omega$ lives on C_{tl}. Here $d^*\omega$ is defined as $(d^*\omega)(\langle xy \rangle) = \sum \omega(p)$, where the sum goes over all plaquettes p such that $\langle xy \rangle \in \partial p$ (I have written Γ as an additive group). Therefore every γ_0 contributing to (6) contains at least a surface S with boundary $\partial S = C_{tl}$. I will call such γ_0 from now on with a slight abuse of notation a surface with boundary C_{tl} as well. Note that the minimal surface $\gamma_{\min}$ with boundary C_{tl} (it consists of all plaquettes in the plane Σ that lie in the interior of C_{tl}) has size $|\gamma_{\min}| = tl$.

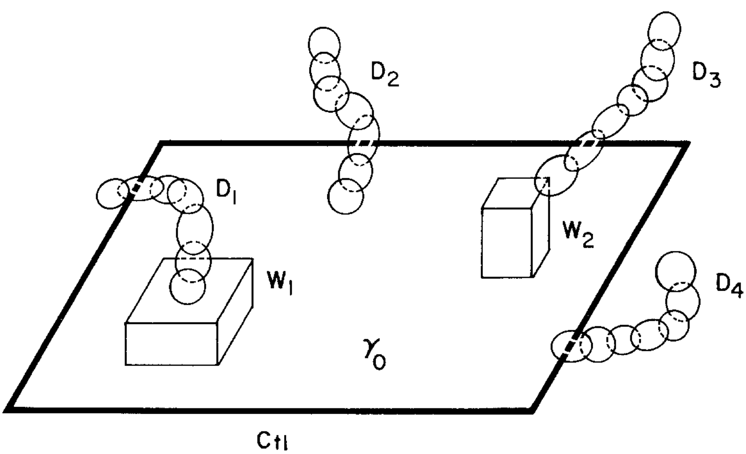

Fig. 1. A decorated surface with boundary C_{tl}. While $D_1,..., D_4$ are decorations of the surface γ_0, the "walls" W_1 and W_2 are part of γ_0.

Next I want to formalize somewhat the notion of a decoration of γ_0. Let X be a function from Γ_0 into the nonnegative integers $\mathbb{Z}_0$ with $n(X) := \sum_{\gamma \in \Gamma_0} X(\gamma) < \infty$. One then calls X a multi-index defined on Γ_0 and sets

$$X! := \prod_{\gamma \in \Gamma_0} X(\gamma)!, \qquad z^X := \prod_{\gamma \in \Gamma_0} z(\gamma)^{X(\gamma)}$$

$$|X| := \sum_{\gamma \in \Gamma_0} X(\gamma)\, |\gamma|$$

where $|\gamma|$ is the number of plaquettes in γ. With a slight abuse of notation I call $\gamma(X) := \cup\{\gamma: X(\gamma) > 0\}$ the support of X. Now X is called a *cluster* if $\gamma(X)$ is a connected set. If in addition $\gamma(X) \cup \gamma_0$ is a connected set (I say X is attached to γ_0), I call X a *decoration of* γ_0. For a multi-index X and a set $\gamma_0 \in \Gamma_0$, I define $\phi_c(\gamma_0, X) := \phi_c(\gamma_0, \gamma_1,..., \gamma_n)$, where I choose $\gamma_1,..., \gamma_n$ to be any sequence that contains exactly $X(\gamma)$ copies of each $\gamma \in \Gamma_0$. For several multi-indices $X_1,..., X_m$ I define

$$\phi_c(\gamma_0, X_1,..., X_m) := \phi_c(\gamma_0, X_1 + \cdots + X_m)$$

Since $\phi_c(\gamma_0, \gamma_1,..., \gamma_n) = 0$ if $\gamma_0 \cup \cdots \cup \gamma_n$ is not a connected set, any sequence $\gamma_1,..., \gamma_n$ contributing to (6) can be decomposed into decorations of γ_0 that are pairwise not connected to each other. Using the fact that

$$\phi_c(\gamma_0, X_1,..., X_m) = \prod_{i=1}^{m} \phi_c(\gamma_0, X_i)$$

if, for $i \neq j$, $\gamma(X_i)$ is not connected to $\gamma(X_j)$ (for a proof see, e.g., Ref. 6, Lemma 3), we can rewrite (6)

$$\langle W_{tl} \rangle = \sum_{\gamma_0} z_C(\gamma_0) \sum_{\{X_1,...,X_m\}}' \prod_{i=1}^{m} z(X \mid \gamma_0) \tag{7}$$

where the sum $\sum'$ goes over the set $\{X_1,..., X_m\}$ of decorations of γ_0 such that $\gamma(X_i)$ and $\gamma(X_j)$ are not connected to each other for $i \neq j$, and

$$z(X \mid \gamma_0) = \frac{z^X}{X!} \phi_c(\gamma_0, X)$$

The reader familiar with Ref. 4 might suggest at this point that we analyze γ_0 using the methods of Dobrushin.[7] One should get a description of γ_0 in terms of pairwise compatible standard walls and it should be possible to regroup the walls corresponding to γ_0 and the decorations of γ_0 into aggregates. The resulting aggregate model then should be analyzed by the methods of Ref. 4.

For this approach it is important, however, that the activity $z(\gamma_0)$ of the surface γ_0 factors into the activity of the minimal surface that can be spanned into C_{tl} times the product of suitably defined activities for the walls of γ_0. But unless G (the gauge group) is Abelian, this property fails for the model considered here. If one considers charged surfaces (to be defined below) with boundary C_{tl}, however, the desired factorization property can be proved to hold.

We therefore continue as follows: Let

$$\rho(\cdot)=\sum_{Q\neq 0}\rho_Q(\cdot),\qquad \rho_Q(\cdot)=\lambda_Q d_Q \chi_Q(\cdot)$$

be the Fourier decomposition of ρ; here $\{\chi_Q(\cdot)\}_{Q\in\hat{G}}$ denotes the set of characters corresponding to the irreducible representations of G, $d_Q=\chi_Q(\mathbb{1})$, and

$$\lambda_Q=(1/\phi_Q)\int\overline{\chi_Q(g)}\,\rho(g)\,dg$$

[note that the trivial representation $0\in\hat{G}$ does not contribute to ρ, due to the normalization condition (4)]. It is then obvious that

$$z_C(\gamma_0)=\sum_{\mathbf{Q}} z_C(\gamma_0,\mathbf{Q}) \tag{8}$$

where the sum goes over functions $\mathbf{Q}\colon\gamma_0\to\hat{G}$ and

$$z_C(\gamma_0,\mathbf{Q})=\int W_{tl}\prod_{p\in\gamma_0}\rho_{Q(p)}(g_p)$$

Inserting (8) into (7), we finally obtain a representation of $\langle W_{tl}\rangle$ as sum over *decorated charged surfaces with boundary* C_{tl},

$$\langle W_{tl}\rangle=\sum_{(\gamma_0,\mathbf{Q})} z_C(\gamma_0,\mathbf{Q})\sum_{\{X_1,\ldots,X_m\}}{}' \prod_{i=1}^{m} z(X\mid\gamma_0) \tag{9}$$

Here I defined a charged surface with boundary C_{tl} as an ordered pair $(\gamma_0,\mathbf{Q})$, where γ_0 is a surface with boundary C_{tl} and $\mathbf{Q}$ is a function $\mathbf{Q}\colon\gamma_0\to\hat{G}$.

To analyze the charged surfaces $(\gamma_0,\mathbf{Q})$ I use the methods of Dobrushin.[7] Rather than describe all the details here, I point out some of the relevant ideas.

Recall that C_{tl} was a loop in a certain plane Σ. Let π be the orthogonal projection from $\mathbb{Z}^d$ onto Σ. I call a set of plaquettes π-connected if its projection is a connected set of plaquettes, links, and points in Σ.

A plaquette p in a charged surface $(\gamma_0, \mathbf{Q})$ is now called *regular* if (i) $\pi(p)$ is a plaquette in the interior of C_{tl} (in particular this implies that p is parallel to Σ), (ii) there is no other plaquette $p' \in \gamma_0$ that has the same projection, and (iii) $Q(p) = q_C$ or $\bar{q}_C$, where q_C is the representation used in the definition of W_{tl} and $\bar{q}_C$ is the conjugate representation [recall that I assumed that the same representation was used in the action; therefore, if β is small enough, $\lambda_{q_C} = \lambda_{\bar{q}_C}$ is the largest Fourier coefficient of $\rho(\cdot)$]. Every plaquette in γ_0 that is not regular is called *excited.*

A pair $(W, \mathbf{Q}')$ is now called[5] a *wall of* $(\gamma_0, \mathbf{Q})$ if W is a π-connectivity component of the set of all excited plaquettes in γ_0 and $\mathbf{Q}'$ is the restriction of $\mathbf{Q}$ onto W. We call $\omega = (W, \mathbf{Q}')$ a *standard wall* if there is a charged surface $(\gamma_0, \mathbf{Q})$ such that ω is the only wall of $(\gamma_0, \mathbf{Q})$. Two standard walls $(W_1 \mathbf{Q}'_2)$ and $(W_2 \mathbf{Q}'_2)$ are called *compatible* if $\pi(W_1)\, \pi(W_2)$ is not a connected set.

It is now possible to translate each wall of a charged surface $(\gamma_0, \mathbf{Q})$ in such a way that the resulting set of walls is a set of pairwise compatible standard walls. It turns out that this correspondence between charged surfaces and sets of pairwise compatible walls is 1–1, and that the activity of a charged surface is equal to the product of suitably defined activities for the standard walls corresponding to $(\gamma_0, \mathbf{Q})$ times the activity of the minimal surface $\gamma_{\min}$ with boundary C_{tl}. Note, however, that it is quite nontrivial to verify these facts for the surfaces considered here. This is mainly due to two complications:

First, the projection $\pi(\gamma)$ lies in general not in the interior of C_{tl}, as it does for the surfaces considered by Dobrushin or by Holický *et al.* This produces boundary effects that are technically hard to analyze. For example, there are now walls $(W, \mathbf{Q}')$ such that $\pi(W)$ is connected, but W is not. This does not occur in the models of Dobrushin or Kotecký.

Second, the above-mentioned factorization property is not obvious in our model, since $z(\gamma_0, \mathbf{Q})$ is given by an integral that in principle couples all the walls of $(\gamma_0, \mathbf{Q})$. Using gauge invariance and a trick proposed by Seiler[8] (see also Ref. 3, pp. 59–62), which involves a lattice gauge theory on a finer lattice in an intermediate step, the desired factorization property can nevertheless be obtained.

Now we can follow the lines of Ref. 4. The idea is to regroup the walls and decorations of a charged surface in such a way that each group has a connected projection, but different groups have projections that are not connected to each other. One then again applies the methods of Dobrushin to these groups (called aggregates in the language of Ref. 4) and finally

[5] For technical reasons a slightly different definition, which implies that all plaquettes in W that are connected to the boundary ∂W of W are still regular, is used in Ref. 3.

obtains a representation of $\langle W_{tl}\rangle$ as the partition function of a certain polymer gas times the activity of the minimal surface $\gamma_{\min}$,

$$\langle W_{tl}\rangle = Z_{tl}^{\mathrm{pol}} z_C(\gamma_{\min}) \tag{10}$$

Note that the free energy density

$$f^{\mathrm{pol}} = -\frac{1}{tl}\log Z_{tl}^{\mathrm{pol}}$$

of the Polymer system is essentially equal to $\tilde{\sigma}(\beta)$, since by the Peter–Weyl theorem

$$z_C(\gamma_{\min}) = \lambda^{tl}$$

where λ is the Fourier coefficient of ρ corresponding to the representation used in the action (and in W_{tl}), and $\lambda(\beta) \to 0$ like β as $\beta \to 0$.

For small β the polymer gas turns out to be dilute, the Mayer expansion for f^{pol} converges, and one can show that f^{pol} is analytic in β. This implies the desired analyticity of $\tilde{\sigma}$ in a neighborhood of zero.

Since the technical details sketched above might obscure the basic simplicity of the proof, I summarize again the main steps: First the standard strong coupling cluster expansion was used to obtain a representation of Wilson loop expectations $\langle W_{tl}\rangle$ as sums over surfaces γ_0 with boundary C_{tl} decorated with clusters of closed surfaces. The techniques of Dobrushin give a description of γ_0 in terms of standard walls. At this point two main new ideas were used to continue: First we used the Fourier expansion of the weight of the geometric surface to obtain a representation of $\langle W_{tl}\rangle$ in terms of charged surfaces. While the desired factorization properties of the weight of the geometric surfaces could not be proven, this could be done for the charged surfaces. Second we regrouped certain walls and decorations into aggregates, and obtained a description of σ in terms of the free energy density of a gas of hard core interacting particles. For strong couplings this gas is dilute, and therefore the Mayer expansion for its free energy density can be used to prove the desired analyticity properties of σ.

I finally want to discuss the condition that the representation q_C used to define W_{tl} represents the center $Z(G)$ in a nontrivial way. Assume that this is not the case. First of all the proof that $z_C(\gamma_0) = 0$ if γ_0 does not contain a surface with boundary C_{tl} breaks down [the condition involving the dual Γ of $Z(G)$ now reads $z_C(\gamma_0) = 0$ unless there is a Γ-valued 2-form ω defined on γ_0 such that $d^*\omega = 0$].

It is known that this is not only a technical problem. Indeed, it can be shown under rather general conditions[1,9] that for $\gamma_0 \in \Gamma_0$ as in Fig. 2, $z_C(\gamma_0) \neq 0$ if q_C is trivial on $Z(G)$. If the representation in the action is the

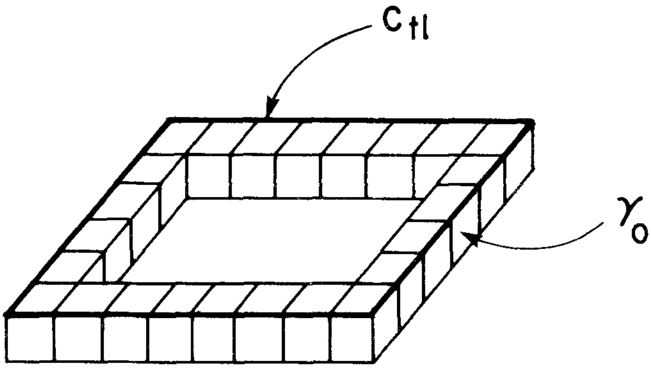

Fig. 2. The torus γ_0.

fundamental representation q_f of $G = U(N)$ or $SU(N)$, and q_C is the adjoint representation q_{adj}, this follows immediately from the fact that $q_{\text{adj}} \in \bar{q}_f \times q_f$. So the leading term in the cluster expansion of $\langle W_{tl} \rangle$ falls off only exponentially with the length $2(t+l)$ of C_{tl}. This suggests the conjecture that for representations q_C that are trivial on $Z(G)$

$$|\langle W_{tl} \rangle| \geq \text{const} \times e^{-\alpha 2(t+l)} \tag{11}$$

with constants const >0 and $\alpha < \infty$, which do not depend on t and l. Note that (11) clearly implies $\sigma = 0$.

It should be possible actually to prove (11) by the methods described in this paper. Instead of excitations over $\gamma_{\min}$, one considers excitations over γ_0 as in Fig. 2. The four ($d = 3$) "ground states" as in Fig. 3 will replace the regular plaquettes, and the polymer gas now lives on C_{tl} instead of $\gamma_{\min}$. Note that there is the additional problem of a certain conservation law, because the "twists" of the ground state must add up to 2π. This problem can be solved by Fourier transformation.[6,10]

In the end one should obtain the existence of the limit

$$\alpha = -\lim_{t,l \to \infty} \frac{1}{2(t+l)} \log \langle W_{tl} \rangle$$

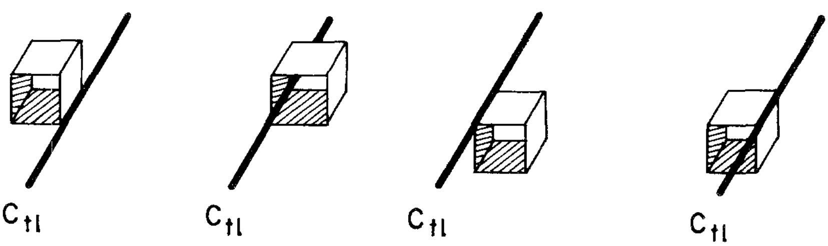

Fig. 3. The four ground states in $d = 3$.

and (for $q_C = q_{\text{adj}}$, and the fundamental representation in the action) the analyticity property

$$\alpha = -4 \log \beta + \tilde{\alpha}(\beta)$$

where $\tilde{\alpha}$ is analytic in some neighborhood of $\beta = 0$.

ACKNOWLEDGMENTS

I thank the organizers of the conference on Statistical Mechanics of Phase Transitions—Mathematical and Physical Aspects (Trebon, 1986) for the pleasant and stimulating atmosphere; I am indebted to Roman Kotecký for his hospitality during a short stay in Prague after the conference. For discussions concerning the subject of this talk I am grateful to many people, in particular Jürg Fröhlich, Erhard Seiler, and Tom Kennedy.

REFERENCES

1. E. Seiler, *Gauge Theories As a Problem of Constructive Field Theory and Statistical mechanics* (Lecture Notes in Physics 159, Springer, Berlin, 1982).
2. G. Münster, High-temperature expansion for the free energy of vortices and the string tension in lattice gauge theories, *Nucl. Phys. B* **180**[FS2]:23 (1981).
3. C. Borgs, Zufallsflächen und Clusterentwicklungen in Gitter–Yang–Mills Theorien (in German), Thesis, University of Munich (1986).
4. P. Holický, R. Kotecký, and M. Zahradník, in preparation.
5. J. K. Wilson, Confinement of quarks, *Phys. Rev. D* **10**:2445 (1974).
6. J. Bricmont and J. Fröhlich, Statistical mechanical methods in particle structure analysis of lattice gauge field theories II, *Commun. Math. Phys.* **98**:553 (1985).
7. R. L. Dobrushin, Gibbs state describing coexistence of phases for a three dimensional Ising model, *Theor. Prob. Appl.* **17**:582 (1972).
8. E. Seiler and C. Borgs, personal communication.
9. E. Seiler, Quantized gauge fields with emphasis on rigorous results, Graduate course, Princeton University (1985).
10. G. Gallavotti, The phase separation line in the two-dimensional Ising model, *Commun. Math. Phys.* **27**:103 (1972).